

# Conserved charges in timelike Warped-AdS<sub>3</sub> spaces

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We consider the timelike version of Warped Anti-de Sitter space (WAdS), which corresponds to the three-dimensional section of the Gödel solution of four-dimensional cosmological Einstein equations. This geometry presents closed timelike curves (CTCs), which are inherited from its four-dimensional embedding. In three dimensions, this type of solutions can be supported without matter provided the graviton acquires mass. Here, among the different ways to consistently give mass to the graviton in three dimensions, we consider the parity-even model known as New Massive Gravity (NMG). In the bulk of timelike WAdS<sub>3</sub> space, we introduce defects that, from the three-dimensional point of view, represent spinning massive particle-like objects. For this type of sources, we investigate the definition of quasi-local gravitational energy as seen from infinity, far beyond the region where the CTCs appear. We also consider the covariant formalism applied to NMG to compute the mass and the angular momentum of spinning particle-like defects, and compare the result with the one obtained by means of the quasi-local stress-tensor. We apply these methods to special limits in which the WAdS<sub>3</sub> solutions coincide with locally AdS<sub>3</sub> and locally AdS<sub>2</sub> × ℝ spaces. Finally, we make some comments about the asymptotic symmetry algebra of asymptotically WAdS<sub>3</sub> spaces in NMG.

## 1 Introduction

In the last years, gravity about three-dimensional Warped Anti-de Sitter (WAdS<sub>3</sub>) spaces has attracted attention due to the fact that it represents one of the most interesting examples of what has been dubbed *non-AdS holography*. Different proposals suggesting

that quantum gravity in WAdS<sub>3</sub> space could be dual to a two-dimensional theory with certain type of conformal invariance have appeared in the literature [1, 2, 3, 4]. It is therefore natural to ask to what extent the holography-inspired techniques to compute observables such as conserved charges can be extended to the case of timelike WAdS<sub>3</sub>. The majority of the works considering WAdS<sub>3</sub> holography in the literature are, however, concerned with the spacelike WAdS<sub>3</sub> spaces. This is because, on the one hand, spacelike spaces can host black holes [5], which are particularly interesting; on the other hand, the fact that stretched timelike WAdS<sub>3</sub> spaces exhibit closed timelike curves (CTCs) is usually regarded as a pathology that makes this case less physically sensible than its squashed spacelike analogue. Nevertheless, there are still good reasons to study the definition of conserved charges in asymptotically timelike WAdS<sub>3</sub> spaces. One such motivation comes from dS/CFT: In dS/CFT, as originally proposed [6], the dual field theory is supposed to be an Euclidean CFT formulated at future infinity. In the static patch, the holographic picture is such that the dual CFT is located beyond the cosmological horizon. Therefore, when trying to apply holographic renormalization techniques, one has to propose a way to define the regularized boundary stress-tensor far beyond the horizon, where the vector that is timelike inside the static patch becomes spacelike. A particular proposal to do so in dS space has been given in Ref. [7], where it was proposed that conserved charges can be defined in terms of the holographic stress-tensor integrating on constant- $t$  codimension-2 surfaces, being  $t$  the coordinate which is timelike inside the static patch. This proposal works well for dS/CFT and seems to be an ingenious trick to deal with backgrounds that do not necessarily admit a globally defined timelike Killing vector. A possible explanation of why the proposal in [7] works has been recently given by Ref. [8], where it has been shown that the dual CFT description does not necessarily have to be placed at the future conformal boundary, but also holds on any fixed timelike slice in the static patch.

Here, we will raise the same kind of questions for timelike WAdS<sub>3</sub> spaces in New Massive Gravity (NMG) [9]. First, we will investigate to what extent we can provide a notion of quasi-local energy in timelike WAdS<sub>3</sub> from far infinity. Secondly, we will study the conserved charges, such as mass and angular momentum of spinning particle-like objects in timelike WAdS<sub>3</sub> using the covariant formalism applied to NMG. We will see that, despite the fact that timelike WAdS<sub>3</sub> exhibits CTCs, this space admits a sensible definition of conserved charges.

The main motivations to study the particular case of NMG are the following: Being a ghost free theory that propagates two degrees of freedom, NMG is the three-dimensional (3D) theory of gravity whose dynamics is closest to that of four-dimensional gravity, in the sense that it propagates two polarizations of a spin-2 particle (instead of zero as in 3D General Relativity, only one as in Topologically Massive Gravity (TMG), or three as in other 3D higher-curvature models). NMG is, on the other hand, the simplest model of three-dimensional gravity that preserves parity (unlike TMG, for instance). This latter property is of particular importance in the case of WAdS<sub>3</sub> backgrounds because it permits to distinguish between the *chiral features* arising from the  $SL(2, \mathbb{R}) \times U(1)$  symmetric

geometry from chiral features inherent of parity-violating effects (which are present in theories such as TMG).

The paper is organized as follows: In Section 2, timelike WAdS<sub>3</sub> space is discussed and its main properties reviewed. In Section 3, we discuss the WAdS<sub>3</sub> spaces as solutions to three-dimensional massive gravity. In Section 4, we propose a definition of quasi-local gravitational energy for defects in asymptotically WAdS<sub>3</sub> spaces. We define the quasi-local stress-tensor for three-dimensional massive gravity and discuss the difficulties encountered when trying to compute both the mass and angular momentum of defects with this method. In addition, we compute the conserved charges associated to defects in timelike WAdS<sub>3</sub> in the covariant formalism adapted to massive gravity. We compare the results obtained for the timelike WAdS<sub>3</sub> defects with the computation of the mass and angular momentum of asymptotically spacelike WAdS<sub>3</sub> black holes. Section 5 contains our conclusions.

## 2 Timelike WAdS<sub>3</sub> space

Timelike WAdS<sub>3</sub> spaces are squashed or stretched deformations of asymptotically three-dimensional Anti-de Sitter spaces (AdS<sub>3</sub>) [10]. In the case of the stretched deformation, WAdS<sub>3</sub> corresponds to the three-dimensional section of the Reboucas-Tiomno one-parameter generalization [11, 12] of the Gödel solution of four-dimensional cosmological Einstein equations, and the existence of CTCs is a property inherited from its four-dimensional ancestor, the Gödel universe [13]. These spaces represent a workable example to address questions such as how to define physically sensible observables, such as conserved charges, in spaces with CTCs.

### 2.1 Timelike WAdS<sub>3</sub> from Gödel metric

Gödel cosmological solution is the direct product of the real line,  $\mathbb{R}$ , and a three-dimensional manifold  $\Sigma$  equipped with a metric [13, 14]

$$ds^2 = - \left( d\hat{t} + e^{\sqrt{2}\omega x} dy \right)^2 + dx^2 + \frac{1}{2} e^{2\sqrt{2}\omega x} dy^2, \quad (1)$$

with coordinates  $x, y, \hat{t} \in \mathbb{R}$ , and  $\omega$  being a real parameter that represents the *vorticity* of the Gödel solution. This coordinate system gives a complete chart of the space, and the four-dimensional solution is then homeomorphic to  $\mathbb{R}^4$ . The space is geodesically complete, and hence singularity free; it is spatially homogeneous, though non-isotropic.

In a convenient system of coordinates, metric (1) above takes the form

$$ds^2 = - \left( dt + \frac{2}{\omega} \sinh^2 \left( \frac{\omega \rho}{\sqrt{2}} \right) d\phi \right)^2 + \frac{1}{2\omega^2} \sinh^2(\sqrt{2}\omega\rho) d\phi^2 + d\rho^2, \quad (2)$$

where the three-dimensional metric is now written as a Hopf fiber over the hyperbolic plane. This space exhibits closed timelike curves, as it can be seen from the role played by coordinates  $t$  and  $\phi$  in the first term of (2).

The prominent properties of the Gödel space persist if one considers a particular one-parameter deformation of the metric (2) which, in particular, permits to interpolate between the three-dimensional section of Gödel space and  $\text{AdS}_3$  [11]. This deformation is given by the metric

$$ds^2 = - \left( dt + \frac{4\omega}{\lambda^2} \sinh^2 \left( \frac{\lambda\rho}{2} \right) d\phi \right)^2 + \frac{\sinh^2(\lambda\rho)}{\lambda^2} d\phi^2 + d\rho^2, \quad (3)$$

which, apart from the vorticity  $\omega$ , includes an additional real parameter  $\lambda$  that controls the deformation. For the particular value  $\lambda^2 = 2\omega^2$ , metric (3) corresponds to the three-dimensional section of Gödel solution (2); when  $\lambda^2 = 4\omega^2$  it corresponds to the universal covering of  $\text{AdS}_3$ . For generic values of  $\lambda$  and  $\omega$  within the range  $0 \leq \lambda^2 \leq 4\omega^2$ , metric (3) describes the timelike stretched WAdS<sub>3</sub> spaces we will be concerned with.

It is convenient to consider a slightly different parameterization: Define the parameter

$$\ell^2 = \frac{2}{\lambda^2 - 2\omega^2}, \quad (4)$$

and then use  $\omega$  and  $\ell^2$  (instead of  $\lambda$ ) to describe the family of WAdS<sub>3</sub> metrics. For instance, in terms of  $\omega$  and  $\ell^2$ , the Gödel solution corresponds to  $\ell^2 = \infty$ , while  $\text{AdS}_3$  space corresponds to  $\ell^2 = \omega^{-2}$ . The range  $0 \leq \lambda^2 \leq 4\omega^2$ , in terms of these parameters, translates into  $|\omega^2\ell^2| \geq 1$ . Notice that  $\omega^2\ell^2$  may take values between  $-1$  and  $-\infty$ . Spaces with  $|\omega^2\ell^2| < 1$  are also interesting, although present a different causal structure; they correspond to the timelike *squashed* WAdS<sub>3</sub> spaces.

Now, continuing with the convenient changes of coordinates, define the new radial variable  $r = 2\lambda^{-2} \sinh^2(\lambda\rho/2)$ , such that  $r \in \mathbb{R}_{\geq 0}$ . Metric (3) now reads

$$ds^2 = -dt^2 - 4\omega r dt d\phi + 2(r + (\ell^{-2} - \omega^2)r^2) d\phi^2 + \frac{dr^2}{2(r + (\ell^{-2} + \omega^2)r^2)}. \quad (5)$$

This is one of the standard ways of representing timelike WAdS<sub>3</sub> space. The curvature invariants associated to this metric are constant, and take the remarkably succinct form

$$R_{\mu_n}^{\mu_1} R_{\mu_1}^{\mu_2} R_{\mu_2}^{\mu_3} \dots R_{\mu_{n-1}}^{\mu_n} = (-1)^n \frac{2^n}{\ell^{2n}} (\omega^{2n} \ell^{2n} + 2). \quad (6)$$

Another interesting property of metric (2) is that it is spatially homogeneous. As it happens with the universal covering of AdS, the WAdS spaces are not globally hyperbolic.

The isometry group of WAdS<sub>3</sub> spaces (5) is  $SL(2, \mathbb{R}) \times U(1)$ , which is generated by four out of the five Killing vectors that Gödel solution admits. This isometry is the remnant piece of the  $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$  isometry group of  $\text{AdS}_3$  that survives through the stretched/squashed deformation.

From (5), it is easy to verify that in the special point  $\omega^2\ell^2 = 1$  the solution tends to  $\text{AdS}_3$  space. Indeed, defining the new coordinates  $\theta = t - \phi$  and  $\rho^2 = 2r$  and replacing  $\omega = \ell = 1$  in (5), gives

$$ds_{\text{AdS}_3}^2 = -(\rho^2 + 1)dt^2 + \frac{d\rho^2}{(\rho^2 + 1)} + \rho^2 d\theta^2. \quad (7)$$

## 2.2 Introducing a defect

Let us now introduce a pointlike defect in spacetime (5). This is achieved by performing the change

$$\phi \rightarrow (1 - \mu)\varphi, \quad \text{with} \quad 0 \leq \mu < 1, \quad (8)$$

while keeping the same periodicity for the  $\varphi$  coordinate, namely  $\varphi \in [0, 2\pi]$ . This certainly changes the global properties of the space in a way that is equivalent to introducing an angular deficit  $\delta\phi = \mu/(2\pi)$  in the original angular coordinate. By doing (8) and rescaling the radial coordinate as  $r \rightarrow r/(1 - \mu)$  one finds the metric

$$\begin{aligned} ds^2 = & -dt^2 - 4\omega r dt d\varphi + 2r \left( (\ell^{-2} - \omega^2)r + (1 - \mu) \right) d\varphi^2 \\ & + \frac{dr^2}{2r ((\omega^2 + \ell^{-2})r + (1 - \mu))}, \end{aligned} \quad (9)$$

where  $t \in \mathbb{R}$ ,  $r \in \mathbb{R}_{\geq 0}$ , and  $\varphi \in [0, 2\pi]$ . This metric shares the asymptotic behavior with (5); namely both have the large  $r$  behavior

$$ds^2 = -dt^2 - 4\omega r dt d\varphi + 2(\ell^{-2} - \omega^2)r^2 d\varphi^2 + \frac{dr^2}{2r^2(\ell^{-2} + \omega^2)} + h_{\mu\nu} dx^\mu dx^\nu, \quad (10)$$

with, in particular,  $\delta g_{\varphi\varphi} \equiv h_{\varphi\varphi} \simeq \mathcal{O}(r)$  and  $\delta g_{rr} \equiv h_{rr} \simeq \mathcal{O}(r^{-3})$ .

Metric (9) represents a particle-like object located at  $r = 0$ , in the bulk of Gödel universe. The object disappears when  $\mu$  tends to zero, which permits to anticipate that  $\mu$  is somehow related to the mass of the defect. More general defects will be introduced later (see (34) below), which will represent spinning point particles in Gödel spacetime.

## 3 Timelike WAdS<sub>3</sub> space in massive gravity

### 3.1 WAdS<sub>3</sub> spaces as gravity backgrounds

A feature that makes WAdS<sub>3</sub> spaces of particular interest is that these geometries appear as exact solutions of a large variety of models, including string theory [15, 16], topologically massive gauge theories [17, 18, 19, 20], higher-derivative theories [21], bi-gravity theories [22], and Einstein gravity non-minimally coupled to matter fields [23]. A minimal setup in which WAdS<sub>3</sub> spaces appear is three-dimensional gravity with no matter fields. Indeed, spacelike and timelike WAdS<sub>3</sub> geometries are exact solutions of pure three-dimensional gravity provided one gives a small mass to the graviton. The graviton mass is what ultimately induces the *vorticity* required to support the Gödel universe or, more precisely, the three-dimensional non-trivial part of it. In three-dimensions, there are different manners to give mass to the graviton in a consistent way. Here, we will adopt the particular parity-even theory of massive gravity proposed in Ref. [9], usually called New Massive Gravity (NMG), which we will review in the next subsection. Our method to compute the quasi-local gravitational energy, in Section 4, amounts to define a boundary stress-tensor for NMG, which is the generalization of the Brown-York quasi-local stress-tensor. For NMG

theory, such a tensor exists and has been defined in Ref. [24]. We will consider such a definition of the quasi-local stress-tensor and use it to compute the mass of the defect in timelike WAdS<sub>3</sub> as seen from infinity, i.e. from the region that is beyond the radius where CTCs appear.

We will first consider a defect in timelike WAdS<sub>3</sub> space, which comes to represent a massive spinless pointlike object. From the four-dimensional point of view, this is like considering a local cosmic string in the Gödel universe. We will propose a physically sensible definition of mass for such a highly-localized source. Intriguingly, the result we will obtain will be shown to account for 1/2 of the Arnowitt-Deser-Misner (ADM) mass of the defect. In addition, the definition of charges in terms of the quasi-local stress-tensor will prove to be not suitable to compute the angular momentum of spinning defects, the failure being associated to the impossibility of regularizing the boundary stress-tensor by means of local counterterms. This will eventually lead us to consider an alternative approach to compute charges. We will consider, in Section 5, the covariant formalism for computing charges in NMG. Let us now introduce the theory.

### 3.2 Three-dimensional New Massive Gravity

In this section, we will discuss asymptotically timelike WAdS<sub>3</sub> spaces in the specific context of three-dimensional NMG. The action of the theory consists of three distinct contributions, namely

$$S = S_{\text{EH}} + S_{\text{NMG}} + S_{\text{B}}, \quad (11)$$

where the first term is the Einstein-Hilbert action

$$S_{\text{EH}} = \frac{1}{16\pi G} \int_{\Sigma} d^3x \sqrt{-g} (\sigma R - 2\Lambda), \quad (12)$$

with  $\sigma = \pm 1$  being a sign that effectively controls the sign of the Newton constant. The second term in (11) is given by

$$S_{\text{NMG}} = \frac{1}{16\pi G m^2} \int_{\Sigma} d^3x \sqrt{-g} \left( R_{\mu\nu} R^{\mu\nu} - \frac{3}{8} R^2 \right), \quad (13)$$

where  $m$  is the mass of the graviton. The third term in (11) is the boundary action, needed for the variational principle to be well-posed. We will discuss the boundary action in the next subsection.

Let us recall the main properties of theory (12)-(13): Around maximally symmetric backgrounds, its linearized limit coincides with the massive spin-2 Fierz-Pauli action, representing a fully covariant extension of the latter. At a generic point of the parameter space  $(\Lambda, m)$ , the theory propagates two massive local degrees of freedom. In addition, NMG admits a rich set of solutions, such as Schrödinger invariant spaces [25], Lifshitz spaces and Lifshitz black holes [26], logarithmic deformation of the Bañados-Teitelboim-Zanelli geometry [27], hairy (A)dS<sub>3</sub> black holes [28], WAdS<sub>3</sub> black holes, and others [21, 29].

The equations of motion derived from (12) and (13) are

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \sigma \Lambda g_{\mu\nu} + \frac{\sigma}{2m^2} K_{\mu\nu} = 0, \quad (14)$$

which, apart from the Einstein tensor, involve the tensor

$$K_{\mu\nu} = 2\square R_{\mu\nu} - \frac{1}{2}\nabla_\mu\nabla_\nu R - \frac{1}{2}\square Rg_{\mu\nu} + 4R_{\mu\alpha\nu\beta}R^{\alpha\beta} - \frac{3}{2}RR_{\mu\nu} - R_{\alpha\beta}R^{\alpha\beta}g_{\mu\nu} + \frac{3}{8}R^2g_{\mu\nu}.$$

Timelike WAdS<sub>3</sub> metrics (9) solve the equations of motion (14) provided the coupling constants satisfy<sup>1</sup>

$$\Lambda = -\frac{(11\omega^4\ell^4 + 28\omega^2\ell^2 - 4)\sigma}{2(19\omega^2\ell^2 - 2)\ell^2}, \quad m^2 = -\frac{(19\omega^2\ell^2 - 2)\sigma}{2\ell^2}. \quad (15)$$

Recall that AdS<sub>3</sub> space corresponds to  $\omega^2\ell^2 = 1$ , for which  $\Lambda = -35\sigma/(34\ell^2)$  and  $m^2 = -17\sigma/(2\ell^2)$ .

### 3.3 Boundary terms

To discuss boundary terms  $S_B$ , let us first rewrite (13) as follows:

$$S_{\text{NMG}} = \frac{1}{16\pi G} \int_{\Sigma} d^3x \sqrt{-g} \left( f^{\mu\nu}(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}) - \frac{1}{4}m^2(f_{\mu\nu}f^{\mu\nu} - f^2) \right). \quad (16)$$

This includes an auxiliary field  $f_{\mu\nu}$ , represented by a rank-2 symmetric tensor. After varying with respect to  $f_{\mu\nu}$ , one finds

$$f_{\mu\nu} = \frac{2}{m^2}(R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu}), \quad (17)$$

which can be plugged back into (16) to reproduce the higher-curvature term (13).

The next step is to consider the ADM type decomposition in the radial direction; that is

$$ds^2 = N^2 dr^2 + \gamma_{ij}(dx^i + N^i dr)(dx^j + N^j dr), \quad (18)$$

where  $N^2$  is the radial analogue of the lapse function,  $N^i$  are the shift functions and  $\gamma_{ij}$  is the two-dimensional metric induced on the constant- $r$  surfaces. The Latin indices  $i, j = 0, 1$ , refer to the coordinates on the constant- $r$  surfaces (namely  $x^0 = t, x^1 = \varphi$ ), while the Greek indices  $\mu, \nu = 0, 1, 2$ , refer to all coordinates, including the radial direction  $x^2 = r$ .

Boundary terms  $S_B$  are introduced in (11) for the variational principle to be defined in such a way that both the metric  $g_{\mu\nu}$  and the auxiliary field  $f_{\mu\nu}$  are fixed on the boundary  $\partial\Sigma$ ; see [24] for details. The boundary action is then given by

$$S_B = -\frac{1}{8\pi G} \int_{\partial\Sigma} d^2x \sqrt{-\gamma} \left( K + \frac{1}{2}\hat{f}^{ij}(K_{ij} - \gamma_{ij}K) \right), \quad (19)$$

where, as said,  $\gamma_{ij}$  is the metric induced on  $\partial\Sigma$ ,  $\gamma = \det(\gamma_{ij})$  and  $K_{ij}$  is the extrinsic curvature, with  $K = \gamma^{ij}K_{ij}$ .  $\hat{f}^{ij}$  in (19) comes from decomposing the auxiliary field  $f^{\mu\nu}$  as follows  $f^{\mu\nu} = \delta_i^\mu\delta_j^\nu f^{ij} + 2\delta_r^\mu\delta_i^\nu h^i + \delta_r^\mu\delta_r^\nu s$  and then defining  $\hat{f}^{ij} \equiv f^{ij} + 2h^{(i}N^{j)} + sN^iN^j$ , and  $\hat{f} \equiv \gamma_{ij}\hat{f}^{ij}$ , where  $a^{(\mu}b^{\nu)} \equiv (a^\mu b^\nu + a^\nu b^\mu)/2$ .

The first term in (19) corresponds to the Gibbons-Hawking term of General Relativity, while the other two terms come from the higher-curvature terms of (16). These terms are preliminary elements to define the boundary stress-tensor, which we will discuss in the next section.

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<sup>1</sup>These solutions persist if one introduces in the equations of motion the Cotton tensor of TMG.

## 4 Conserved charges

### 4.1 The quasi-local stress-tensor

The Brown-York quasi-local stress-tensor  $T_{ij}$  is obtained by varying action (11) with respect to the metric  $\gamma^{ij}$ , [30]. That is,

$$T_{ij} = \frac{2}{\sqrt{-\gamma}} \frac{\delta S}{\delta \gamma^{ij}}|_{r=\text{const}}, \quad (20)$$

which yields [24]

$$\begin{aligned} T^{ij} = & \frac{1}{8\pi G} (K^{ij} - K\gamma^{ij}) - \frac{1}{8\pi G} \left( \frac{1}{2} \hat{f} K^{ij} + \nabla^{(i} \hat{h}^{j)} - \frac{1}{2} \nabla_r \hat{f}^{ij} + K_k^{(i} \hat{f}^{j)k} \right. \\ & \left. - \frac{1}{2} N^2 s K^{ij} - \gamma^{ij} (\nabla_k \hat{h}^k - \frac{1}{2} N^2 s K + \frac{1}{2} \hat{f} K - \frac{1}{2} \nabla_r \hat{f}) \right), \end{aligned} \quad (21)$$

where  $\hat{h}^i = N(h^i + sN^i N^j)$ . The covariant  $r$ -derivative  $\nabla_r$  acting on  $\hat{f}^{ij}$  is defined as follows

$$\nabla_r \hat{f}^{ij} = \frac{1}{N} \left( \partial_r \hat{f}^{ij} - N^k \partial_k \hat{f}^{ij} + 2\hat{f}^{k(i} \partial_k N^{j)} \right), \quad \nabla_r \hat{f} = \frac{1}{N} \left( \partial_r \hat{f} - N^k \partial_k \hat{f} \right). \quad (22)$$

When taking the limit  $r \rightarrow \infty$  in the definition (20), stress-tensor (21) is found to diverge. Without a proper regularization procedure, this would result in an infinite value for the conserved charges. To solve this problem, one may try to improve the definition (20) by including additional boundary terms to the action, provided such terms do not spoil the variational principle. In Ref. [31], this method was applied to the case of spacelike WAdS<sub>3</sub>. It was shown that, despite the persistent divergences of some components of  $T_{ij}$ , adding a boundary cosmological constant term to  $S_B$  makes the functional action finite and yields a finite quasi-local energy. We can try to do the same here for the timelike case and improve the stress-tensor (20) by adding a piece

$$T_{ij} \rightarrow T_{ij} - \frac{\zeta}{8\pi G} \gamma_{ij}, \quad (23)$$

which would come from a boundary contribution

$$S_B \rightarrow S_B + \frac{\zeta}{8\pi G} \int d^2 x \sqrt{-\gamma}, \quad (24)$$

where  $\zeta$  is a coefficient fixed by requiring the action to be finite. The value of this coefficient is found to be

$$\zeta = -\frac{\sigma 8\omega^2 \ell \sqrt{2(\omega^2 \ell^2 + 1)}}{(19\omega^2 \ell^2 - 2)}. \quad (25)$$

### 4.2 Quasi-local gravitational energy

The boundary stress-tensor (20), once improved by the adding of (23), yields the definition of conserved charges  $Q_{\tilde{\xi}}$ , associated to vectors  $\tilde{\xi}$  that generate isometries on  $\partial\Sigma$ . These *boundary Killing vectors*  $\tilde{\xi}$  are defined by the equation

$$\mathcal{L}_{\tilde{\xi}} \gamma_{ij} = 0, \quad (26)$$

for the induced metric. Then, the charges are defined by integrating the projection of the boundary stress-tensor on the vector  $\tilde{\xi}$  and a unitary vector  $u$  that is orthogonal to the constant- $t$  surfaces. That is,

$$Q_{\tilde{\xi}} = \int d\varphi \varrho u^i T_{ij} \tilde{\xi}^j, \quad (27)$$

where  $\varrho$  is given by the induced metric written in the form

$$d\Sigma^2 = -N_\Sigma^2 dt^2 + \varrho^2 (dt + N_\Sigma^\varphi d\varphi)^2. \quad (28)$$

In particular, for the WAdS<sub>3</sub> defects we have

$$\begin{aligned} \varrho^2 &= 2(1-\mu)r + 2(\ell^{-2} - \omega^2)r^2, \\ N_\Sigma^\varphi &= -\frac{\omega r}{(1-\mu)r + (\ell^{-2} - \omega^2)r^2}, \\ N_\Sigma^2 &= 1 + \frac{2\omega^2 r^2}{(1-\mu)r + (\ell^{-2} - \omega^2)r^2}. \end{aligned} \quad (29)$$

With these ingredients, we are ready to compute the mass of the defects: The unitary vector orthogonal to the constant- $t$  surfaces is given by  $u = -N_\Sigma(r)dt$ . Considering a timelike boundary Killing vector

$$\tilde{\xi}^i = N_\Sigma u^i, \quad (30)$$

(i.e. timelike in the region where the source is located) we find a value for the quasi-local energy  $\mathcal{M} = Q_{\tilde{\xi}}$ , which reads

$$\mathcal{M} = \frac{2\sigma\omega^2\ell^2(\mu-1)}{(19\omega^2\ell^2-2)G} = \frac{2(\mu-1)}{19G} \left( \sigma - \frac{1}{m^2\ell^2} \right), \quad (31)$$

where we used that  $2m^2\ell^2\sigma = 2 - 19\omega^2\ell^2$ .

Let us first compare the result (31) with the special case of locally AdS<sub>3</sub> solutions, which correspond to  $\omega^2\ell^2 = 1$ . In this case, (31) reduces to

$$\mathcal{M}_{\omega^2\ell^2=1} = \frac{2\sigma(\mu-1)}{17G}, \quad (32)$$

and, indeed, this is seen to match the mass of a defect in locally AdS<sub>3</sub> space in NMG. To see this explicitly, let us be reminded of the fact that in the case of NMG in AdS<sub>3</sub> the mass of a deficit angle (a particular case of the BTZ geometry) is given by [32]

$$\mathcal{M}_{\text{AdS}_3} = \frac{(\mu-1)}{8G} \left( \sigma + \frac{1}{2m^2\ell^2} \right) = \frac{2\sigma(\mu-1)}{17G}, \quad (33)$$

where we used that  $\omega^2\ell^2 = 1$  precisely corresponds to  $2m^2\ell^2\sigma = -17$ . That is, (31) reduces to the value (32) at that point of the parameter space. In principle, we could be tempted to take this matching as a consistency check of the result (31). However, if we think of it carefully, we conclude that there is a priori no good reason to expect (31) to coincide with (33) in the  $\omega^2\ell^2 \rightarrow 1$  limit. This is because, even when in that limit WAdS<sub>3</sub> space becomes AdS<sub>3</sub> space, the latter shows up in a coordinate system which is not the one usually considered when computing the ADM charges of BTZ geometry. This is similar to

what happens in the case of asymptotically WAdS<sub>3</sub> black holes, whose conserved charges, as functions of the horizons radii, do not tend to the charges of BTZ black holes in the  $\nu \rightarrow 1$  limit (being  $\nu$  the parameter that controls the deformation in that case; see the conventions in [1]). In fact, we will see in the next section that the correct value of the gravitational mass associated to a pointlike defect in timelike WAdS<sub>3</sub> space coincides with (31) only up to a factor of 1/2. This feature has already been observed in the context of spacelike WAdS<sub>3</sub> solutions [31].

As it happens with spacelike WAdS<sub>3</sub> spaces, the method of computing charges using the quasi-local stress-tensor (20) does not suffice to give a finite result for the angular momentum of spinning defects. This is basically because there seems to be no manner to regularize all the components of (20) by means of local boundary counterterms. This means that, in order to study spinning defects, it is necessary to consider a different method for computing conserved charges. With this motivation, we will consider in the following section the covariant formalism.

### 4.3 Covariant formalism in New Massive Gravity

Let us now consider spinning defects. The metric of Gödel spacetime with both mass and angular momentum reads <sup>2</sup>

$$ds^2 = -dt^2 - 4\omega r dt d\varphi + \frac{dr^2}{(2r^2\omega^2 + \lambda_{\mu,j}(r))} - (2r^2\omega^2 - \lambda_{\mu,j}(r)) d\varphi^2, \quad (34)$$

where

$$\lambda_{\mu,j}(r) = \frac{2r^2}{\ell^2} + 2(1-\mu)r - j\ell^2, \quad (35)$$

and where  $t \in \mathbb{R}$ ,  $r \in \mathbb{R}_{\geq 0}$ ,  $0 \leq \mu \leq 1$ , and  $\phi \in [0, 2\pi]$ . Metric (34) involves a new parameter  $j \in \mathbb{R}$ , and reduces to (9) when  $j = 0$ . Notice also that only  $\xi^t \sim \partial_t$  and  $\xi^\varphi \sim \partial_\varphi$  out of the four generators of  $SL(2, \mathbb{R}) \times U(1)$  survive as exact Killing vectors of the metric (34).

Such as in the case of the parameter  $\mu$ , the introduction of  $j$  is achieved by means of a (improper, i.e. not globally well-defined) diffeomorphism from metric (5). Metric (34) solves the equations of motion (14) for the parameters (15).

In the covariant formalism [33, 34], conserved charges associated to an asymptotic Killing vector  $\xi$  are given in three spacetime dimensions by the expression

$$\delta Q_\xi[\delta g, g] = \frac{1}{16\pi G} \int_0^{2\pi} \sqrt{-g} \epsilon_{\mu\nu\varphi} k_\xi^{\mu\nu} [\delta g, g] d\varphi, \quad (36)$$

with  $g$  a solution,  $\delta g$  a linearized perturbation around it, and  $k_\xi^{\mu\nu}[\delta g, g]$  being a one-form potential of the linearized theory. In [35], this potential was computed for exact Killing vectors in NMG using the so-called Abbott-Deser-Tekin (ADT) formalism. The result can be written

$$k_\xi^{\mu\nu} = Q_R^{\mu\nu} + \frac{1}{2m^2} Q_K^{\mu\nu}, \quad (37)$$

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<sup>2</sup>Notice that we can assign dimensions to the parameters and coordinates as follows:  $[t] = l^1$ ,  $[r] = l^2$ ,  $[\phi] = l^0$ ,  $[\ell] = l^1$ ,  $[\omega] = l^{-1}$ ,  $[\mu] = l^0$ ,  $[j] = l^0$ , where  $l$  has dimension of length.

where the first contribution comes from the pure GR part of the equations of motion, while  $Q_K^{\mu\nu}$  accounts for the contribution of the  $K_{\mu\nu}$  tensor of NMG, whose explicit expression can be found in equations (22), (28) and (29) in [35], respectively.

#### 4.4 Mass and angular momentum in the covariant formalism

One can use (37) and plug it into (36) to compute the (variation of the) mass and angular momentum, for which the Killing vectors are, respectively,  $\partial_t$  and  $\partial_\varphi$ . This procedure has been implemented in a Mathematica code and, for  $\sigma = 1$ , gives<sup>3</sup>

$$\mathcal{M} = \frac{4(\mu - 1)\ell^2\omega^2}{G(19\ell^2\omega^2 - 2)}, \quad (38)$$

and

$$\mathcal{J} = -\frac{4j\ell^4\omega^3}{G(19\ell^2\omega^2 - 2)}, \quad (39)$$

for the mass and the angular momentum of the solution (34), respectively. Notice that, as expected, the angular momentum changes its sign when  $\omega$  does so.

Expressions (38), (39) are the correct values of the conserved charges. Intriguingly, the Brown-York quasi-local energy obtained in (31) gives only one half of the mass<sup>4</sup>.

A special case to consider is the actual Gödel spacetime, which corresponds to the limit  $\ell \rightarrow \infty$ . In this case, the mass formula (38) yields

$$\mathcal{M}_{\text{Göd}} = \frac{4(\mu - 1)}{19G}, \quad (40)$$

which is independent of  $\omega$ . For  $\mu = 0$  the result is negative and is of crucial importance in the study of the spacelike WAdS<sub>3</sub> black hole spectrum [3, 36].

Another special case to analyze is the AdS<sub>2</sub> × ℝ space. This corresponds to the limit  $\omega \rightarrow 0$ . To see this explicitly, we define coordinate  $\tilde{\rho}^2 = 1 + 4(r^2/\ell^4 + r/\ell^2)$ , in which the metric for  $\omega = 0$  takes the form

$$ds_{|\omega=0}^2 = -dt^2 + ds_{\text{AdS}_2}^2 = -dt^2 + \frac{\ell^2}{2}(\tilde{\rho}^2 - 1)d\phi^2 + \frac{\ell^2}{2}\frac{d\tilde{\rho}^2}{(\tilde{\rho}^2 - 1)}. \quad (41)$$

In this case, the mass also tends to zero,

$$\mathcal{M}_{\mathbb{R} \times \text{AdS}_2} = 0. \quad (42)$$

Locally AdS<sub>2</sub> × ℝ spaces appear in the limit in which (15) yields  $\Lambda = -m^2$  [28].

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<sup>3</sup>This result is up to  $\mu$ -independent and  $j$ -independent terms, which can not be gathered in the integration.

<sup>4</sup>The same phenomenon has been observed in the case of spacelike black holes, where the quasi-local energy gives one half of the black hole mass computed by covariant methods, the latter being the value fulfilling the first principle [35].

## 5 Conclusions

In this paper, we have investigated the definition of conserved charges in timelike WAdS<sub>3</sub> spacetimes, which exhibit CTCs. We have considered these spaces in the context of NMG. Timelike WAdS<sub>3</sub> spacetimes in NMG represent a workable example to address questions such as how to define physically sensible observables, such as conserved charges, in spaces that do not possess a globally defined timelike Killing vector.

For stretched and squashed timelike WAdS<sub>3</sub> spaces, we have investigated several features related to the feasibility of defining conserved charges. One of the questions we have addressed was how to provide a sensible definition of quasi-local gravitational energy in these spacetimes that exhibit CTCs. The motivation for doing this was studying to what extent the holography-inspired methods can be applied to this example of non-AdS holography. We have succeeded in doing this for non-spinning defects. However, the difficulties encountered when trying to adapt this method to spinning solutions eventually led us to consider an alternative way of computing charges. We have resorted to the covariant formalism applied to NMG, which was shown to be suitable to compute the mass and angular momentum of a more general type of defects that represent spinning particle-like objects in the bulk of WAdS<sub>3</sub>.

The question remains as to whether it is possible to formulate a holographic renormalization recipe in WAdS<sub>3</sub> spaces. The obstruction encountered when trying to do this in Section 4 was the impossibility of regularizing the full boundary stress-tensor in terms of local boundary counterterms. This phenomenon had also been observed both in TMG and in NMG for the case of spacelike WAdS<sub>3</sub>, suggesting this is a general feature of this type of backgrounds. Whether or not this problem is related to the lack of Lorentz invariance in the dual theory is still to be understood.

Before concluding, let us mention that the covariant method for computing charges discussed in this paper can be adapted to the case of charges associated to asymptotic isometries. In a companion paper [36], it will be shown that the algebra of charges in asymptotically WAdS<sub>3</sub> spaces is given by an infinite-dimensional algebra that coincides with the semidirect sum of Virasoro algebra with non-vanishing central charge and an affine  $\hat{u}(1)_k$  Kač-Moody algebra.

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## Appendix: Relation with the WAdS<sub>3</sub> black holes

As we will see, the relation between timelike charges we have obtained and the mass and angular momentum of the so-called Warped black holes (WBH) is not as simple as one could a priori think. WBHs are black hole solutions that asymptote stretched spacelike WAdS<sub>3</sub> space; see [1] and references therein. As we will describe below, these black holes can be obtained from the timelike solution by means of a complex change of coordinates: Consider first the double Wick rotation

$$t \rightarrow i\tau, \quad \varphi \rightarrow -i\Theta, \quad \omega \rightarrow -\omega, \quad r \rightarrow -r, \quad j \rightarrow -j, \quad (43)$$

and, secondly,  $\tau = t' - \ell\sqrt{j}\Theta$ . Finally, in order to compare with the coordinates used in the literature, let us rescale time as  $t' \rightarrow LT$ .

The change of coordinates above maps the timelike metric (34) into the WBH solution<sup>5</sup>

$$\begin{aligned} ds^2 = & L^2 dT^2 + \frac{L^2 dR^2}{(\nu^2 + 3)(R - r_+)(R - r_-)} + L^2(2\nu R - \sqrt{r_+ r_- (\nu^2 + 3)})dTd\Theta \\ & + \frac{RL^2}{4} \left[ 3(\nu^2 - 1)R + (\nu^2 + 3)(r_+ + r_-) - 4\nu\sqrt{r_+ r_- (\nu^2 + 3)} \right] d\Theta^2, \end{aligned} \quad (44)$$

with  $R = -2r/L^2$  and provided one identifies the parameters as follows

$$\begin{aligned} \nu &= \omega L; \quad L^2 = \frac{3}{\omega^2 + 2\ell^{-2}}; \\ r_{\pm} &= \frac{\ell^2}{L^2} \left[ \frac{-(1 - \mu) \pm \sqrt{(1 - \mu)^2 - 2(\omega^2 \ell^2 + 1)j}}{(\omega^2 \ell^2 + 1)} \right]. \end{aligned} \quad (45)$$

Notice the useful relations

$$r_+ + r_- = \frac{2\ell^2(\mu - 1)}{L^2(1 + \ell^2\omega^2)}; \quad r_+ r_- = \frac{2j\ell^4}{L^4(1 + \ell^2\omega^2)}. \quad (46)$$

The timelike and spacelike Killing vectors are related in the following way

$$\partial_t = \frac{i}{L}\partial_T, \quad \partial_\varphi = \frac{\ell}{L}\sqrt{j}\partial_T + i\partial_\Theta. \quad (47)$$

This charge dependent change of coordinates makes the relation between timelike and spacelike charges more involved than a mere analytic continuation.

Changing in (44)  $LT \rightarrow t$ ,  $R \rightarrow r$  and  $L\Theta \rightarrow \varphi$ , we can assign the dimensions as  $[t] = l^1$ ,  $[r] = l^1$ ,  $[\varphi] = l^0$ ,  $[L] = l^1$ ,  $[\nu] = l^0$ ,  $[r_{\pm}] = l^1$  and the expression of the mass of the

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<sup>5</sup>See Eq. (4.1) in Ref. [1].

WBH then becomes<sup>6</sup>

$$\mathcal{M}_{\text{WBH}} = Q_{\partial_T} = \frac{\nu(\nu^2 + 3)}{GL(20\nu^2 - 3)} \left( (r_- + r_+)\nu - \sqrt{r_+r_-(\nu^2 + 3)} \right), \quad (48)$$

while the expression for the angular momentum is<sup>7</sup>

$$\mathcal{J}_{\text{WBH}} = Q_{\partial_\Theta} = \frac{\nu(\nu^2 + 3)}{4GL(20\nu^2 - 3)} \left( (5\nu^2 + 3)r_+r_- - 2\nu\sqrt{r_+r_-(\nu^2 + 3)}(r_+ + r_-) \right). \quad (49)$$

Using the relations (45) between the spacelike and timelike parameters, one observes that going from the timelike to the spacelike metric involves a charge-dependent and globally not-well defined change of coordinates, namely the definition  $\tau = t' - \ell\sqrt{j}\Theta$  above. This implies that the spacelike and timelike charges do not coincide. Only in the case  $j = 0$ , one sees that the masses are related<sup>8</sup> according to  $\partial_t \sim L^{-1}\partial_T$ ,

$$\mathcal{M}_{\text{WBH}}|_{j=0} = L^{-1}\mathcal{M}. \quad (50)$$

It is important to remark that, in the case of spinning defects in timelike WAdS<sub>3</sub>, and due to the  $j$ -dependent change of coordinates, the conserved charges can not be simply obtained from the mass and angular momentum of spacelike solutions.

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<sup>6</sup>This expression comes from (D.4) in [35], which coincides with (27) in [31] without the extra factor 1/2 which should be absent. Note that this expression has also been obtained independently with the covariant formalism.

<sup>7</sup>Result taken from (30) in [31] which has been crossed checked with [21]. Note that this expression has also been obtained independently in the covariant formalism.

<sup>8</sup>Up to a  $\mu$ -independent factor, which can not be seen in the integration.

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